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ORIGINAL PAPER

Uniform convergence and computation of the generalized exponential integrals

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Abstract In this paper, uniform convergence of the generalized exponential (GE) integrals is investigated. We also study the continuity, differentiability, summability and asymptotic behaviour of GE integrals. Then we give an accurate and efficient computation algorithm for these integrals.

Keywords Generalized exponential integrals · Uniform convergence · Radiative transfer · Computational algorithm

1 Introduction

The generalized exponential (GE) integrals play an important role in various fields of theoretical physics, quantum chemistry, theory of transport process, theory of fluid flow and astrophysics [1–9]. These integrals are especially useful for the study of multiple light scattering in a multidimensional medium with scattering phase function and multidimensional radiative transfer problems [10–14]. Note that the GE integrals have been studied by numerous authors with different algorithms [15–19].

The GE integrals are defined as

$$G_k(x) = \frac{1}{(k-1)!} \int_1^{\infty} e^{-xy} (\ln y)^{k-1} \frac{dy}{y}, \tag{1}$$

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where $k = 1, 2, \ldots$ Milgram type generalization of the GE integral has the form [20]:

$$E_s^j(x) = \frac{1}{\Gamma(j+1)} \int_1^\infty e^{-xy} (\ln y)^j \frac{dy}{y^s}, \tag{2}$$

where $s \in (-\infty, \infty)$, $j \in (-1, \infty)$ and $\Gamma(j+1)$ is the Gamma function. It follows from (1) and (2) that, $E_1^{k-1}(x) = G_k(x)$. If $j = 0$, then we obtain the well known exponential integral E_s :

$$E_s(x) = \int_1^\infty e^{-xy} \frac{dy}{y^s}. \tag{3}$$

This integral is frequently used in the theory of transport processes, theory of fluid flow and radiative transfer problems from astrophysics. In [16], the authors obtain, for the GE integrals E_s^n, the series expansion formulas in terms of binomial and multinomial coefficients:

$$\begin{aligned} E_s^n(x) &= \frac{1}{n!} \sum_{n_1=0}^{n} \sum_{n_2=0}^{n-n_1} \sum_{n_3=0}^{n-n_1-n_2} \cdots \sum_{n_t=0}^{n-n_1-n_2-\ldots-n_{t-1}} \\ &\times \sum_{k=0}^{n_1+2n_2+3n_3+\cdots+tn_t} (-1)^k F_{n_1,n_2,\ldots,n_t}(n) \\ &\times F_k(n_1+2n_2+3n_3+\cdots+tn_t) \frac{1}{2^{n_2} 3^{n_3} \ldots t^{n_t}} E_{k+s}(x), \end{aligned} \tag{4}$$

where $t = 1, 2, \ldots$, $F_k(n) = \frac{n!}{k!(n-k)!}$ are binomial coefficients and

$$F_{n_1,n_2,\ldots,n_t}(n) = F_{n_1}(n) F_{n_2}(n-n_1) \ldots F_{n_t}(n-n_1-n_2-\cdots-n_{t-1})$$

are multinomial coefficients. By using (4), the authors gave numerical approximations for GE integrals E_s^n.

The exponential integrals $E_n (n = 1, 2, \ldots)$ were also calculated by the recurrence formula [19]:

$$(n-1)E_n(x) = e^{-x} - xE_{n-1}(x), \quad 0 < x < 1,$$

where the starting term is given by

$$E_1(x) = \int_1^\infty y^{-1} e^{-xy} dy.$$

Note that, the continuity, integrability, asymptotic and other properties of GE integrals depend on the uniform convergence of these integrals. But, in literature, the theory of uniform convergence of GE integrals are not investigated so far. In this work, we study on the convergence properties of GE integrals. We also investigate the continuity, integrability and asymyptotic behaviour of GE integrals. Then, we give an algorithm for efficient numerical approximation.

2 Uniform convergence

Let us consider the improper integral

$$\int_1^\infty f(x,y)g(x,y)dy, \quad x \in I. \tag{5}$$

We will use the following uniformly convergent test [21]:

Theorem 1 *Let the functions* $f, g, \frac{\partial g}{\partial y}$ *are continuous with respect to* y *in* $[1,\infty)$ *and* $\frac{\partial g}{\partial y}$ *has the same sign for all* $x \in I$ *and* $y \in [1,\infty)$. *If*

$$\lim_{y\to\infty} g(x,y) = 0$$

uniformly with respect to $x \in I$, *and the function defined by*

$$\Phi(x,t) = \int_1^t f(x,y)dy$$

is bounded in the domain $D = \{(x,t) : x \in I,\ t \in [1,\infty)\}$, *then the integral given by* (5) *is uniformly convergent with respect to* x *in* I.

Now we investigate the uniform convergence of GE integrals. We shall begin with the exponential integral E_s.

Proposition 2 (i) *If* $s \in (-\infty, 1]$, *then the exponential integral* (3) *is uniformly convergent with respect to* x *on* $[\varepsilon,\infty)$, $\varepsilon > 0$.

(ii) *For* $s \in (-\infty, 1]$, *the exponential integral* (3) *is nonuniformly convergent with respect to* x *on* $[0,\infty)$.

(iii) *If* $s \in (1,\infty)$, *then the exponential integral* (3) *is uniformly convergent with respect to* x *on* $[0,\infty)$.

Proof (i). We consider the cases $s \in (-\infty, 0]$ and $s \in (0, 1]$, separately. Let $s \in (-\infty, 1]$ and consider the functions

$$f_1(x,y) = e^{-\frac{x}{2}y}y^{-s},\ \ g_1(x,y) = e^{-\frac{x}{2}y},\ \ x \in I = [\varepsilon,\infty),\ \varepsilon > 0.$$

It is clear that, the functions $f_1, g_1, \frac{\partial g_1}{\partial y}$ are continuous in y and the sign of $\frac{\partial g_1}{\partial y}$ is -1 for all $x \in [\varepsilon, \infty)$, $y \in [1, \infty)$. Moreover,

$$\lim_{y\to\infty} g_1(x, y) = \lim_{y\to\infty} e^{-\frac{x}{2}y} = 0$$

uniformly with respect to $x \in [\varepsilon, \infty)$, $\varepsilon > 0$. Let

$$\Phi(x, t) = \int_1^t f_1(x, y)dy = \int_1^t e^{-\frac{x}{2}y} y^{-s} dy,\ x \in [\varepsilon, \infty), y \in [1, \infty).$$

We have

$$\begin{aligned}
\Phi(x, t) &= \int_1^t e^{-\frac{x}{2}y} y^{-s} dy \leq \int_1^\infty e^{-\frac{x}{2}y} y^{-s} dy \\
&\leq \left(\frac{\varepsilon}{2}\right)^{s-1} \int_{\varepsilon/2}^\infty e^{-u} u^{-s} du \\
&\leq \left(\frac{\varepsilon}{2}\right)^{s-1} \int_0^\infty e^{-u} u^{-s} du \\
&= \left(\frac{\varepsilon}{2}\right)^{s-1} \Gamma(1-s),\ \ x \in [\varepsilon, \infty),\ t \in [1, \infty),
\end{aligned}$$

which means that the function $\Phi(x, t)$ is bounded for all $x \in [\varepsilon, \infty)$, $t \in [1, \infty)$.Thus, if $s \in (-\infty, 0]$, then the exponential integral

$$E_s(x) = \int_1^\infty e^{-xy} \frac{dy}{y^s} = \int_1^\infty f_1(x, y) g_1(x, y) dy$$

is uniformly convergent with respect to x on $x \in [\varepsilon, \infty)$, $\varepsilon > 0$. Now we assume that $s \in (0, 1]$. Let, this time,

$$f_2(x, y) = e^{-xy},\ \ g_2(x, y) = \frac{1}{y^s},\ \ x \in I = [\varepsilon, \infty),\ \varepsilon > 0.$$

Then, in a similar way, we can show that the exponential integral (3) is uniformly convergent with respect x on $[\varepsilon, \infty)$, $\varepsilon > 0$.

(ii). For $s \in (-\infty, 1]$, we obtain

$$E_s(x) = \int_1^\infty e^{-xy} \frac{dy}{y^s} = \frac{1}{x} \int_x^\infty e^{-t} \frac{x^s}{t^s} dt$$
$$= x^{s-1} \int_x^\infty t^{-s} e^{-t} dt \to \infty \quad \text{for } x \to 0.$$

Therefore, for $s \in (-\infty, 1]$, the exponential integral E_s is nonuniformly convergent with respect to x on $[0, \infty)$.
(iii). If $s \in (1, \infty)$, then we find that

$$e^{-xy} \frac{1}{y^s} \leq \frac{1}{y^s}, \ x \in [0, \infty).$$

Since,

$$\int_1^\infty \frac{1}{y^s} dy < \infty,$$

we see that, by Weierstrass test [22], the integral (3) is uniformly convergent with respect to x on $[0, \infty)$. □

By the Proposition 2, we obtain the following properties of the exponential integral:

Corollary 3 (i) *If* $s \in (-\infty, 1]$, *then the function* $E_s(x)$ *is continuous on* $[\varepsilon, \infty), \varepsilon > 0$.
(ii) *In the case* $s \in (1, \infty)$, *the function* $E_s(x)$ *is continuous on* $[0, \infty)$.
(iii) *For all* $s \in (-\infty, \infty)$ *and* $x \in [\varepsilon, \infty), \varepsilon > 0$,

$$\frac{d^k}{dx^k} E_s(x) = (-1)^k E_{s-k}(x), \quad k = 1, 2, \ldots$$

holds.
(iv) *If* $s \in (1, \infty)$, *then* $E_s(x) \in L_1(0, \infty)$, *where*

$$L_1(0, \infty) := \left\{ \varphi : \int_0^\infty |\varphi(x)| \, dx < \infty \right\}.$$

(v) *For all* $s \in (-\infty, \infty)$, *the* $x-$*axis is the asymptote of the function* E_s.

Proof The properties (i)-(iii) can be shown easily. We prove (iv) and (v):
If $s \in (1, \infty)$, then we get

$$\begin{aligned}\int_0^\infty |E_s(x)|\,dx &= \int_0^\infty E_s(x)dx \\ &= \int_0^\infty \int_1^\infty e^{-xy}\frac{dy}{y^s}dx = \int_1^\infty \frac{1}{y^s}\int_0^\infty e^{-xy}dxdy \\ &= \int_1^\infty \frac{1}{y^{s+1}}dy = \frac{1}{s} < \infty.\end{aligned}$$

By the uniform convergence of E_s for $s \in (-\infty, \infty)$, we find that

$$\lim_{x\to\infty} E_s(x) = \lim_{x\to\infty}\int_1^\infty e^{-xy}\frac{dy}{y^s} = \int_1^\infty \frac{1}{y^s}\lim_{x\to\infty} e^{-xy}dy = 0.$$

Thus, the $x-$axis is the asymptote of $E_s(x)$ □

Now we consider the GE integral $E_s^j(x)$ defined by

$$E_s^j(x) = \frac{1}{\Gamma(j+1)}\int_1^\infty e^{-xy}(\ln y)^j\frac{dy}{y^s},$$

where $s \in (-\infty, \infty)$, $j \in (-1, \infty)$.

Proposition 4 (i) *For $-\infty < s - j \leq 1$, the GE integral $E_s^j(x)$ is uniformly convergent with respect to x on $[\varepsilon, \infty)$, $\varepsilon > 0$.*

(ii) *If $1 < s - j < \infty$, the GE integral $E_s^j(x)$ is uniformly convergent with respect to x on $[0, \infty)$.*

Proof By using the inequality $\ln(1+t) \leq t$, $t \in [0, \infty)$, then we get

$$\ln y = \ln[1 + (y-1)] \leq y - 1 < y, \;\; y \in [1, \infty). \tag{6}$$

Thus, we obtain

$$\begin{aligned} E_s^j(x) &= \frac{1}{\Gamma(j+1)} \int_1^\infty e^{-xy} (\ln y)^j \frac{dy}{y^s} \\ &\leq \frac{1}{\Gamma(j+1)} \int_1^\infty e^{-xy} y^j \frac{dy}{y^s} \\ &= \frac{1}{\Gamma(j+1)} E_{s-j}(x), \end{aligned}$$

by (6). Thus, from the inequality

$$E_s^j(x) \leq \frac{1}{\Gamma(j+1)} E_{s-j}(x)$$

and from the Proposition 2, we immediately get (i) and (ii). □

In a similar way of Corollary 3, we have;

Corollary 5 (i) *For* $-\infty < s - j \leq 1$, *the function* $E_s^j(x)$ *is continuous on* $[\varepsilon, \infty)$, $\varepsilon > 0$.

(ii) *For the case* $1 < s - j < \infty$, *the function* $E_s^j(x)$ *is continuous on* $[0, \infty)$.

(iii) *For all* $s \in (-\infty, \infty)$ $j \in (-1, \infty)$ *and* $x \in [\varepsilon, \infty)$, $\varepsilon > 0$,

$$\frac{d^k}{dx^k} E_s^j(x) = (-1)^k E_{s-k}^j(x), \quad k = 1, 2, \ldots \tag{7}$$

holds.

(iv) *If* $1 < s - j < \infty$, *then* $E_s^j(x) \in L_1(0, \infty)$.

(v) For all $s \in (-\infty, \infty)$, $j \in (-1, \infty)$, the $x-$axis is the asymyptote of the function E_s^j.

It is clear that

$$G_k(x) = E_1^{k-1}(x), \; k = 1, 2, \ldots. \tag{8}$$

From (8), we obtain the following:

Proposition 6 (i) *For all* $k = 1, 2, \ldots$, *the GE integral* G_k *is uniformly convergent with respect to* x *and continuous on* $[\varepsilon, \infty)$, $\varepsilon > 0$.

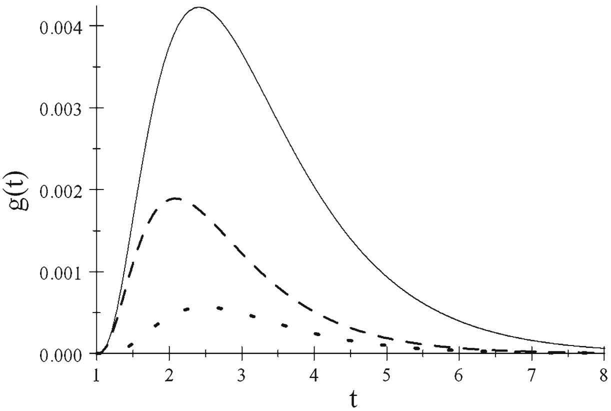

Fig. 1 The graph of the integrand $g(t)$ for some values of j, s, and x

(ii) *For all* $k, m = 1, 2, \ldots$ *and* $x \in [\varepsilon, \infty)$, $\varepsilon > 0$,

$$\frac{d^m}{dx^m} G_k(x) = (-1)^m E_{1-m}^{k-1}(x).$$

(iii) *The* $x-$*axis is the asymptote of the function* G_k.

For any fixed values of j, s and x, $E_s^j(x)$ correspond to the area under the curve given by the function

$$g(t) = \frac{e^{-xt}(\ln t)^j}{\Gamma(j+1)t^s},\ 1 \leq t < \infty.$$

A typical graph of $g(t)$ is shown in Fig. 1.

3 Numerical algorithm

The numerical computation for some GE integrals have been studied by several authors. Those computation methods consist of multinomial series or asymptotic series (see [16, and references cited there]) or Chebyshev polinomial expansion for $G_k(x)$ which include mass computations [15]. Here, we give a simple and "cheap", yet an accurate algorithm to compute the GE integrals even in a moderate PC. The basic trick here we use is to map the infinite range of integration to a finite one:

By letting $t = e^{-y}$ in

$$E_s^j(x) = \frac{1}{\Gamma(j+1)} \int_1^\infty e^{-xy} (\ln y)^j \frac{dy}{y^s},$$

we get the equivalent definite integral

$$E_s^j(x) = \frac{1}{\Gamma(j+1)} \int\limits_0^{1/e} t^{x-1} \frac{[\ln(-\ln t)]^j}{(-\ln t)^s} dt. \tag{9}$$

There are several methods for computing the integral (9). To get rid of the integrand to be evaluated at left endpoint, giving possible computation problems, and to get a fast and efficient computation, we use Euler-Maclaurin formula

$$\int\limits_a^b f(t)dt = \frac{h}{2} \sum_{i=1}^{n} (f_i + f_{i+1}) + O(h^2),$$

an extended midpoint rule [23], as a good choice and a Romberg algorithm by tripling the step size driving from it, since the GE integral is uniformly convergent. Here

$$f(t) = t^{x-1} \frac{[\ln(-\ln t)]^j}{(-\ln t)^s},$$

$f_i = f(t_i)$, $h = (b-a)/(n-1)$, $t_i = a + ih$ $(1 \leq i \leq n-1)$. The computational algorithm is as follows:

```
eps=10e-20 //  the error tolerance
b=exp(-1.)
s_old=0
for n=1 to maks do
   if n=1 then
      i=1
      s=b*f(0.5*b) // the crudest estimate for the integral
  else do
   // subsequent values of n increases the accuracy
  // by adding 2 * 3^(n-2) additional interior points
      m=i
      d=b/(3*m)
      dd=2*d
      t=0.5*d
      sum=0.0
          for k=1 to m do
              sum=sum+f(t)
              t=t+dd
              sum=sum+f(t)
              t=t+d
          end k loop
      i=i*3
      s=(s+b*sum/m)/3. // new sum is added to the old integral
    end if              // to give new approximation
    s=s/Gamma(j+1)
    if |s_old-s|<eps then stop //s is the integral value
    else s_sold=s
    end if
 end n loop
integral=s/Gamma(j+1)  // final estimate of the integral
```

Table 1 Some test values of $E_s^j(x)$ computed with maks = 20 in the algorithm

s	j	x	$E_s^j(x)$	Ref. [16]	Ref. [15]
5	0	1	7.0454237448E-02		
5	0	5	7.0576069309E-04		
1	0.5	1	1.5088706697E-01		
1	3	0.1	1.0230239779E-00		
2	3	0	3.0889122834E-01		
1	3	1	1.1070895444E-02	1.1020742057E-02	1.1070954E-02
1	3	2	7.7405116516E-04	7.7362544708E-04	7.740512E-04
1	3	5	2.9027143901E-06	2.9027148235E-06	2.9028E-06
1	3	10	2.0929700728E-09	2.0931348611E-09	2.1E-09
3	2	4.5	2.3796918445E-05	2.3796883709E-05	
5	3	7.4	2.1887383236E-08	2.1889513585E-08	
10	3	3.8	6.5852647902E-07	6.5860177925E-07	
12	2	12.3	3.1541832914E-10	3.1541832921E-10	

All decimal places given in column 3 are exact

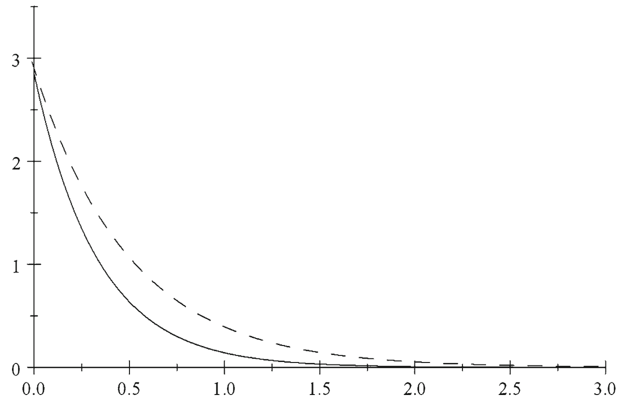

Fig. 2 Graph of $E_s^j(x)$ for any fixed s and j

In computation of $\Gamma(j+1)$, we use the algorithm given in [24]. For several values of j, s and x, computed values with 20 digit precision of the GE integral $E_s^j(x)$ are given and compared to the approximations given by [16] and [15] in Table 1.

From the definition of $E_s^j(x)$ and from the relation (7), we conclude that for fixed values of j and s, $E_s^j(x)$ is positive valued, decreasing and upward convex function. Figure 2 shows a typical graph of the GE integral for two different values of j and s.

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